The Girsanov Multiplier
A Case Study

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In this white paper we will use the Girsanov Multiplier to move the mean of a standardized normal distribution. A standardized normal distribution has mean zero and variance one. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with using the Girsanov Multiplier to change the distribution of stock price from Measure P, which is the actual probability distribution, to Measure Q, which is the risk-neutral probability distribution. The goal is to develop an equation for random stock price such that expected stock price at time t discounted at the risk-free rate is a martingale. The table below presents the model assumptions...

Table 1: Model Assumptions

Description	Value	Notes
Asset price at time zero	100.00	Price in dollars
Annual return mean (%)	15.00	Discrete-time rate
Annual return volatility (%)	35.00	Discrete-time rate
Annual distribution rate (%)	5.00	Discrete-time rate
Risk-free interest rate (%)	4.00	Discrete-time rate

Question 1: What is expected stock price at the end of year 5 under Measure P?

Question 2: What is expected stock price at the end of year 5 under Measure Q?

Question 3: What is the mean and variance of a random variable under Measure Q?

Expected Stock Price Under Measure P

We will define the variable μ to be the continuous-time expected rate of return, the variable ω to be the continuous-time distribution rate, and the variable σ to be the continuous-time return volatility. Because the conversion of discrete-time volatility to the continuous-time equivalent is problematic we will assume that the two volatility measures are equal. Using the parameters in Table 1 above the equations for these variables are...

$$\mu = \ln\left(1 + 0.15\right) = 0.1398 \text{ ...and... } \omega = \ln\left(1 + 0.05\right) = 0.0488 \text{ ...and... } \sigma = 0.3500 \tag{1}$$

We will define the variable λ as follows...

$$\lambda = \mu - \omega - \frac{1}{2}\sigma^2 \tag{2}$$

We will define the variable S_t to be random stock price at time t, the variable σ to be return volatility, and the variable δW_t to be the change in the driving brownian motion at time t. Using Equation (2) above and the parameters in Table 1 above the stochastic differential equation for the change in stock price at time t under the actual probability Measure P is...

$$\delta S_t = \lambda\, S_t\, \delta t + \sigma\, S_t\, \delta W_t \text{ ...where... } \delta W_t \sim \left[0, \delta t\right] \tag{3}$$

The solution to Equation (3) above is the equation for stock price at time t under Measure P, which is...

$$S_t = S_0 \,\text{Exp}\left\{\lambda\, t + \sigma \sqrt{t}\, Z\right\} \text{ ...where... } Z \sim \left[0, 1\right] \tag{4}$$

Using Equation (4) above the equation for expected stock price at time t under the actual probability Measure P is...

$$\mathbb{E}^P\left[S_t\right] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\, \text{Exp}\left\{-\frac{1}{2}\, Z^2\right\} S_0 \,\text{Exp}\left\{\lambda\, t + \sigma \sqrt{t}\, Z\right\} \delta Z \tag{5}$$

Note that we can rewrite Equation (4) above to be a function of the normally-distributed random variable θ. The alternate equation for stock price is...

$$S_t = S_0 \,\text{Exp}\left\{\theta_t\right\} \text{ ...where... mean of } \theta_t = \lambda\, t \text{ ...and... variance of } \theta_t = \sigma^2 t \tag{6}$$

Per Equations (5) and (6) above stock price is lognormally-distributed, which means that the solution to expected stock price under the actual probability Measure P is... [1]

$$\mathbb{E}^P\left[S_t\right] = S_0 \,\text{Exp}\left\{\text{return mean} + \frac{1}{2}\,\text{return variance}\right] = S_0 \,\text{Exp}\left\{\left(\lambda + \frac{1}{2}\,\sigma^2\right) t\right\} \tag{7}$$

Using Equation (2) above we can rewrite Equation (7) above as...

$$\mathbb{E}^P\left[S_t\right] = S_0 \,\text{Exp}\left\{\left(\mu - \omega - \frac{1}{2}\,\sigma^2 + \frac{1}{2}\,\sigma^2\right) t\right\} = S_0 \,\text{Exp}\left\{\left(\mu - \omega\right) t\right\} \tag{8}$$

Expected Stock Price Under Measure Q

We will define the variable α to be the continuous-time risk-free rate. Using the parameters in Table 1 above the equation for α is...

$$\alpha = \ln\left(1 + 0.04\right) = 0.0392 \tag{9}$$

Under the risk-neutral Measure Q all assets earn the risk-free rate. Using Equations (1) and (9) above the equation for expected stock price at time t under the risk-neutral Measure Q is...

$$\mathbb{E}^Q\left[S_t\right] = S_0 \,\text{Exp}\left\{\left(\alpha - \omega\right) t\right\} \tag{10}$$

We will define the function $g(Z)$ to be the Girsanov multiplier at time t. Using Equation (4) above we can write Equation (10) above as...

$$\mathbb{E}^Q\left[S_t\right] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\, \text{Exp}\left\{-\frac{1}{2}\, Z^2\right\} S_0 \,\text{Exp}\left\{\lambda\, t + \sigma \sqrt{t}\, Z\right\} g(Z)\, \delta Z \tag{11}$$

The base equation for the function $g(Z)$ that moves the mean of a normal distribution from m to n is... [2]

$$g(Z) = \text{Exp}\left\{\frac{n - m}{v}\, Z - \frac{n^2 - m^2}{2\, v}\right\} \text{ ...where... } Z \sim N\left[m, v\right] \text{ ...and... } n = \text{new mean} \tag{12}$$

Given that the random variable Z in Equation (11) above is normally-distributed with mean zero and variance one (i.e. a standardized normal distribution) then the variables m and v in Equation (12) above are equal to zero and one, respectively. We can therefore rewrite Equation (12) above as...

$$g(Z) = \text{Exp}\left\{n\, Z - \frac{1}{2}\, n^2\right\} \text{ ...where... } Z \sim N\left[0, 1\right] \text{ ...and... } n = \text{new mean} \tag{13}$$

Using Equations (10) and (13) above we can rewrite Equation (11) above as...

$$S_0 \,\mathrm{Exp}\left\{\left(\alpha-\omega\right)t\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}S_0\,\mathrm{Exp}\left\{\lambda\,t+\sigma\,\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{n\,Z-\frac{1}{2}\,n^2\right\}\delta Z \tag{14}$$

Using Appendix Equation (25) below the value of the variable n in Equation (14) above is...

$$n=\left(\alpha-\mu\right)t\Big/\sigma\,\sqrt{t} \tag{15}$$

The Answers To Our Hypothetical Problem

Question 1: What is expected stock price at the end of year 5 under Measure P?

Using Equations (1), (2) and (8) above and the parameters from Table 1 above the answer to the question is...

$$\mathbb{E}^P\left[S_5\right]=S_0\,\mathrm{Exp}\left\{\left(\mu-\omega\right)t\right\}=100.00\times\mathrm{Exp}\left\{\left(0.1398-0.0488\right)\times 5.00\right\}=157.62 \tag{16}$$

Question 2: What is expected stock price at the end of year 5 under Measure Q?

Using Equations (9) and (14) above and the parameters from Table 1 above the answer to the question is...

$$S_0\,\mathrm{Exp}\left\{\left(\alpha-\omega\right)t\right\}=100.00\times\mathrm{Exp}\left\{\left(0.0392-0.0488\right)\times 5.00\right\}=95.31 \tag{17}$$

Question 3: What is the mean and variance of a random variable under Measure Q?

Using Equations (1), (9) and (15) above the mean of a random variable under Measure Q is...

$$n=\left(\alpha-\mu\right)t\Big/\sigma\,\sqrt{t}=\left(0.0392-0.1398\right)\times 5.00\Big/\left(0.3500\times\sqrt{5.00}\right)=-0.6427 \tag{18}$$

The variance of the random variable is unchanged (i.e. variance is one).

References

[1] Gary Schurman, *The Lognormal Distribution,* June, 2015

[2] Gary Schurman, *The Girasnov Multiplier - Base Equation,* May, 2017

Appendix

A. We want to solve for the variable n in Equation (14) above. Noting that using Equation (10) above we can rewrite Equation (14) above as...

$$S_0 \,\mathrm{Exp}\left\{\left(\alpha-\omega\right)t\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}S_0\,\mathrm{Exp}\left\{\lambda\,t+\sigma\,\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{n\,Z-\frac{1}{2}\,n^2\right\}\delta Z \tag{19}$$

We can rewrite Equation (19) above as...

$$S_0\,\mathrm{Exp}\left\{\left(\alpha-\omega\right)t\right\}=S_0\,\mathrm{Exp}\left\{\lambda\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z^2-2\,n\,Z-2\,\sigma\,\sqrt{t}\,Z+n^2\Big)\right\}\delta Z$$

$$\mathrm{Exp}\left\{\left(\alpha-\omega-\lambda\right)t\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z^2-2\,n\,Z-2\,\sigma\,\sqrt{t}\,Z+n^2\Big)\right\}\delta Z \tag{20}$$

We will make the following definitions...

$$\theta = Z - n - \sigma\sqrt{t} \;\text{...such that...}\; \theta^2 = Z^2 - 2\,n\,Z - 2\,\sigma\sqrt{t}\,Z + n^2 + 2\,n\,\sigma\sqrt{t} + \sigma^2 t \;\text{...and...}\; \delta\theta = \delta Z \tag{21}$$

Using the definitions in Equation (21) above we can rewrite Equation (20) above as...

$$\begin{aligned}
\text{Exp}\left\{\left(\alpha - \omega - \lambda\right)t\right\} &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\left(\theta^2 - 2\,n\,\sigma\sqrt{t} - \sigma^2\,t\right)\right\}\delta\theta \\
\text{Exp}\left\{\left(\alpha - \omega - \lambda\right)t\right\} &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\text{Exp}\left\{n\,\sigma\sqrt{t} + \frac{1}{2}\,\sigma^2\,t\right\}\delta\theta \\
\text{Exp}\left\{\left(\alpha - \omega - \lambda\right)t\right\}\text{Exp}\left\{-n\,\sigma\sqrt{t} - \frac{1}{2}\,\sigma^2\,t\right\} &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta \\
\text{Exp}\left\{\left(\alpha - \omega - \lambda - \frac{1}{2}\,\sigma^2\right)t - n\,\sigma\sqrt{t}\right\} &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta
\end{aligned} \tag{22}$$

Noting that the integral on the left hand side of Equation (22) above is equal to one we can rewrite that equation as...

$$\text{Exp}\left\{\left(\alpha - \omega - \lambda - \frac{1}{2}\,\sigma^2\right)t - n\,\sigma\sqrt{t}\right\} = 1 \tag{23}$$

If we take the log of both sides of Equation (23) above then we can solve for the value of the variable n, which is...

$$\begin{aligned}
\left(\alpha - \omega - \lambda - \frac{1}{2}\,\sigma^2\right)t - n\,\sigma\sqrt{t} &= 0 \\
\left(\alpha - \omega - \lambda - \frac{1}{2}\,\sigma^2\right)t &= n\,\sigma\sqrt{t} \\
\left(\alpha - \omega - \lambda - \frac{1}{2}\,\sigma^2\right)t \Big/ \sigma\sqrt{t} &= n
\end{aligned} \tag{24}$$

Using Equation (2) above we can rewrite Equation (24) above as...

$$n = \left(\alpha - \omega - \left(\mu - \omega - \frac{1}{2}\,\sigma^2\right) - \frac{1}{2}\,\sigma^2\right)t \Big/ \sigma\sqrt{t} = \left(\alpha - \mu\right)t \Big/ \sigma\sqrt{t} \tag{25}$$